hep-ph/9502342
DTP/95/16
February 1995

Fermion Number Violation in the Background of a Gauge Field in Minkowski Space

Valentin V. Khoze ♠

Department of Physics, Science Laboratories
University of Durham, Durham, DH1 3LE, United Kingdom

Abstract

Anomalous fermion number violation is studied in the background of a pure $SU(2)$ gauge field in Minkowski space using the method of N. Christ. It is demonstrated that the chiral fermion number is violated by at most an integer amount. Then the method is applied for a spherically symmetric Minkowski space classical gauge field in the background. These classical gauge fields are finite energy solutions to pure $SU(2)$ equations of motion with in general non-integer topological charge. We show that in the classical background which during a finite time-interval matches such solutions the fermion number violation is integer and non-zero. In particular, we calculate the violation of the fermion number in the presence of Lüscher-Schechter solutions. The meaning of anomaly equation and applications to QCD and electroweak theory are briefly discussed. We also comment on the relation of the results of this paper to the previous work.

♠ E-mail: valya.khoze@durham.ac.uk

1. Introduction

In the Standard Model fermion number is not conserved [1] since gauge field configurations with non-zero topological charge Q,

$$Q = \frac{g^2}{16\pi^2} \int d^4x \; \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}\mathrm{tr}(F_{\mu\nu}F_{\alpha\beta}) \;, \tag{1.1}$$

cause violation of conservation laws due to anomalies [2]. In fact, the fermion number current-density $\hat{j}^\mu = \hat{\bar{\psi}}_{\mathrm{L}}\gamma_\mu\hat{\psi}_{\mathrm{L}}$ for each left-handed fermion flavour is not conserved according to the anomaly equation,

$$\partial_\mu \hat{j}^\mu = \frac{g^2}{16\pi^2}\frac{1}{2}\epsilon^{\mu\nu\alpha\beta}\mathrm{tr}(F_{\mu\nu}F_{\alpha\beta}) \;. \tag{1.2}$$

In a non-Abelian gauge theory there exist an unstable finite energy static solution called the sphaleron [3]. It is a saddle-point of the gauge field potential energy and its energy, E_{sp}, is the barrier height between the different vacuum sectors of the theory. When the transition between two different sectors occurs, the topological charge Q of the gauge field interpolating between two different sectors is non-zero and, according to the anomaly, eq. (1.2), the fermion number changes. One way of understanding these effects is to use semiclassical barrier penetration approach where the tunneling solutions are Euclidean instantons [4]. In the electroweak theory the height of the barrier E_{sp} is of order $M_{\mathrm{w}}/\alpha_{\mathrm{w}} \sim 10\mathrm{TeV}$ and at energies much below this, fermion number violation is exponentially suppressed. It has been suggested that fermion number violating processes may become unsuppressed in the scattering processes at sufficiently high energies [5]. An intuitive way to put it is that with increasing energy the field should tunnel under smaller and smaller portion of the barrier and at the energy higher than the barrier height instead of tunneling through the barrier the field configuration passes over it. Passage over the sphaleron barrier is classically allowed and should be mediated by a classical solution in Minkowski space-time. This is in contrast with the tunneling process which is dominated by a classical solution in Euclidean or even more generally complex time (for references and see a review [6]).

Minkowski space-time approach to fermion number violation may be separated into three parts: the creation of finite energy gauge field configurations by particle collisions, their classical evolution with time and the dynamics of fermion number (or chirality)

violation in the presence of such classical background gauge sources. The classical evolution of certain gauge field configurations in Minkowski space was addressed in Refs. [7-8]. In this paper we study fermion number violation in the background of a gauge field in Minkowski space. Then we apply our results to the case of spherically symmetric classical gauge configurations in the background.

In the Euclidean approach the functional integral is dominated by instantons or instanton-like configurations with finite action. These configurations fall into homotopy classes and require the topological charge Q to be an *integer*. Then according to the anomaly equation (1.2), the number of fermions of each flavour is changed [1] by an integer amount Q.

On classical solutions in Minkowski space Q in general can take any value, not just an integer [7,8]. This is a consequence of the fact that classical Minkowski gauge fields do not approach just some pure gauges in the far past and future, but the finite energy radiation is always present. In this case when the topological charge of a classical gauge field background is not an integer one may ask what is the anomalous fermion production in such a background. This is the motivation of the present paper.

The outline of the paper is as follows. In Section 2 we review the method of N. Christ [9] of studying fermion number violation in a class of background gauge fields. Then we derive a formula for a fermion number violation in such backgrounds and show that it is always an integer. In Section 3 we first review results of Farhi, Khoze and Singleton [7] on classical Yang-Mills system in the spherical ansatz. Then the approach of the Section 2 is applied to the background gauge configurations which match classical solutions of Ref. [7] at all times t except the early past, $t \to T_{\min}$, and the far future, $t \to T_{\max}$. At these times we switch off the gauge invariant degrees of freedom of the background field. This should correspond to the physical situation of interest where an initial coherent gauge field configuration was produced in the course of quantum collision at some early time, $T_{\min}$, and then evolved classically before decaying into quantum radiation at some late time, $T_{\max}$. The idea of our work is to calculate the violation of the fermion number which occurred during the classical evolution of the initial coherent state before it decayed. We assume here that there were no fermion number violation before the coherent field was created or after it decayed. It will be seen that the fermion number violation which occurs

in classical backgrounds is independent on the way of how the interaction is switched off at early and late times and neither it depends on the times $T_{\min}$ and $T_{\max}$ as far as their absolute values are much greater than some characteristic time-scale, $|t_*|$, associated with the solutions. Thus, the fermion number in our approach is indeed violated only during the classical evolution of the initial coherent configuration and not at the moment of its creation or decay. Moreover, we will see that it occurs at the instant where the background field passes over some sphaleron-like configuration. We will calculate the fermion number violation in the presence of classical solutions explicitly and demonstrate that it is integer and in general non-zero.

The further questions which arise from this work are discussed in Section 4.

2. Fermion Number Violation in the Background Gauge Field

In this Section we present our interpretation of the approach of N. Christ, Ref. [9], Section IV C. Then we will demonstrate that the fermion number is violated by at most an *integer* amount for a general class of gauge field backgrounds.

For simplicity we consider the case of a single left-handed fermion flavour $\hat{\psi}_{\rm L} \equiv \frac{1}{2}(1-\gamma_5)\,\hat{\psi}$ coupled to an external $SU(2)$ gauge field. The generalization for the fermion content of the realistic theory is straightforward. The Fermion operator $\hat{\psi}_{\rm L}$ obeys

$$i\gamma^\mu\left(\partial_\mu - igA_\mu\right)\hat{\psi}_{\rm L} = 0\ . \tag{2.1}$$

From now on we will suppress the L-subscript of the Fermi-fields bearing in mind that all $\hat{\psi}$-s are left-handed. The hats distinguish the operator-valued fields from the c-numbers.

We are interested here in background fields $A_\mu(\mathbf{x},t)$ which in the early past, $T_{\min} < t < T_{\rm i} \ll 0$, and in the far future, $0 \ll T_{\rm f} < t < T_{\max}$, can be cast in the following form:

$$A_\mu(\mathbf{x}, t < T_{\rm i}) = U_{\rm in}(\mathbf{x})\big[\frac{i}{g}\partial_\mu + B^{\rm in}_\mu(\mathbf{x},t)\big]U^\dagger_{\rm in}(\mathbf{x})\ , \tag{2.2a}$$

$$A_\mu(\mathbf{x}, t > T_{\rm f}) = U_{\rm out}(\mathbf{x})\big[\frac{i}{g}\partial_\mu + B^{\rm out}_\mu(\mathbf{x},t)\big]U^\dagger_{\rm out}(\mathbf{x})\ . \tag{2.2b}$$

Here $U_{\rm in}(\mathbf{x})$ and $U_{\rm out}(\mathbf{x})$ are $SU(2)$-valued continuous functions of $\mathbf{x}$ which, as $\mathbf{x}\to\infty$, approach direction-independent constants. Thus, $U_{\rm in}(\mathbf{x})$ and $U_{\rm out}(\mathbf{x})$ can be characterized

by winding numbers, $\nu[U_{\rm in}]$ and $\nu[U_{\rm out}]$ which are integer numbers. The gauge fields $B^{\rm in}_\mu(\mathbf{x},t)$ and $B^{\rm out}_\mu(\mathbf{x},t)$ on the right hand side of eqs. (2.2) are required to have essentially *finite* support in the $\mathbf{x}$-space at any fixed time t and to vanish at any $\mathbf{x}$ as time goes respectively to $T_{\rm min}$ or $T_{\rm max}$.

We want to study a process of creation of fermions in the background gauge field specified above. Since we are concerned with particle creation it is important to be able to distinguish a positive energy mode from a negative energy mode and in general in order to count particles we would like to have descrete energy levels. For this reason we compactify the $\mathbf{x}$-space at spatial infinity at any *fixed* time t. This compactification is not in contradiction with the gauge field backgrounds we consider, since $U_{\rm in}(\mathbf{x})$ and $U_{\rm out}(\mathbf{x})$ approach direction-independent constants at spatial infinity and $B^{\rm in}_\mu(\mathbf{x},t)$ and $B^{\rm out}_\mu(\mathbf{x},t)$ are zero at spatial infinity at a fixed time because of the essentially finite support requirement. In other words, when we'll have to deal with the order of limits in Minkowski space-time, our prescription will always be to first let the spatial variable $\mathbf{x}$ go to (compactified) infinity and then (if needed) to let the time $T_{\rm min} < t < T_{\rm max}$ go to the infinite past, $T_{\rm min} \to -\infty$, or infinite future, $T_{\rm max} \to +\infty$.

Our program now is to first find the Fermi-operator $\hat{\psi}(\mathbf{x},t)$. Then we can construct the fermion number operator and consider its expectation values at $t = T_{\rm min}$ and $t = T_{\rm max}$. The difference between these expectation values will give the fermion number violation.

The Fermi-operator $\hat{\psi}(\mathbf{x},t)$ is obtained by the procedure of the second quantization from the c-number general solution to the equation of motion (2.1). To obtain this we have to find a complete set of c-number solutions to (2.1).

We first consider a c-number solution of

$$i\gamma^\mu\left(\partial_\mu - igA_\mu(\mathbf{x},t)\right)\psi(\mathbf{x},t) = 0 \ . \tag{2.3}$$

Let us make a gauge transformation with the gauge function $U = U_{\rm in}(\mathbf{x})$ of eq. (2.2a),

$$A_\mu(\mathbf{x},t) = U_{\rm in}(\mathbf{x})\big[\frac{i}{g}\partial_\mu + B^{\rm in}_\mu(\mathbf{x},t)\big]U^\dagger_{\rm in}(\mathbf{x}) \ , \tag{2.4a}$$

$$\psi(\mathbf{x},t) = U_{\rm in}(\mathbf{x})\zeta(\mathbf{x},t) \ . \tag{2.4b}$$

In terms of new variables $B^{\rm in}_\mu(\mathbf{x},t)$ and $\zeta(\mathbf{x},t)$ eq. (2.3) reads

$$i\gamma^\mu\left(\partial_\mu - igB^{\rm in}_\mu(\mathbf{x},t)\right)\zeta(\mathbf{x},t) = 0 \ . \tag{2.5}$$

Equations (2.4) describe the gauge in which the background gauge field is $B_\mu^{\rm in}$ which has to vanish at $t = T_{\rm min}$. Thus, in this gauge fermions become free in the early past.

There is a complete orthonormal set of solutions of eq. (2.5) which we call an in-set, $\{\zeta_n^{\rm in\pm}(\mathbf{x}, t)\}_{n=1}^{\infty}$, with the initial condition that

$$\zeta_n^{\rm in\pm}(\mathbf{x}, t) \to \psi_n^{\pm}(\mathbf{x})\, e^{\mp iE_n t} \quad \text{as} \quad t \to T_{\rm min} \quad . \tag{2.6}$$

Here $\psi_n^{\pm}(\mathbf{x})$ are positive and negative energy eigenfunctions of the free Dirac Hamiltonian,

$$-i\boldsymbol{\alpha} \cdot \boldsymbol{\nabla} \psi_n^{\pm}(\mathbf{x}) = \pm E_n \psi_n^{\pm}(\mathbf{x}) \quad , \tag{2.7}$$

where $\boldsymbol{\alpha} = \gamma^0 \boldsymbol{\gamma}$ and $E_n > 0$. (By a judicious choice of (compactified) boundary conditions for fermion fields at spatial infinity one can make each negative energy equal to minus a positive energy and also eliminate the zero energy eigenvalue.)

Equation (2.5) can be cast in the retarded Yang-Feldman form:

$$\zeta_n^{\rm in\pm}(\mathbf{x}, t) = \psi_n^{\pm}(\mathbf{x})\, e^{\mp iE_n t} \; - \; g \int_{T_{\rm min}}^{t} dy_0 \int d\mathbf{y} \Delta^{\rm ret}(x-y)\; \gamma^\mu B_\mu^{\rm in}(\mathbf{y}, y_0)\; \zeta_n^{\rm in\pm}(\mathbf{y}, y_0) \; , \tag{2.8}$$

where the first term on the right hand side is the solution of the free Dirac equation and $\Delta^{\rm ret}(x-y)$ is the retarded Green function,

$$i\gamma^\mu \partial_\mu \; \Delta^{\rm ret}(x-y) = \delta^{(4)}(x-y) \; , \tag{2.9a}$$

$$\Delta^{\rm ret}(x-y) \sim \theta(x_0 - y_0) \; . \tag{2.9b}$$

The (retarded) initial condition (2.6) is satisfied only for such backgrounds $B_\mu^{\rm in}$ that the integral on the right hand side of eq. (2.8) vanishes as $t \to T_{\rm min}$. If the equation (2.8) can be solved by iterations, the in-set elements are given by the perturbative formula:

$$\begin{aligned} \zeta_n^{\rm in\pm}(\mathbf{x}, t) &= \psi_n^{\pm}(\mathbf{x})\, e^{\mp iE_n t} \\ &- g \int_{T_{\rm min}}^{t} dy_0 \int d\mathbf{y} \Delta^{\rm ret}(x-y)\; \gamma^\mu B_\mu^{\rm in}(\mathbf{y}, y_0)\; \psi_n^{\pm}(\mathbf{y})\, e^{\mp iE_n y_0} \; + \; \dots \; . \end{aligned} \tag{2.10}$$

We can now finally return to our specification of $B_\mu^{\rm in}$ in the beginning of the Section: $B_\mu^{\rm in}$ is required to vanish as $t \to T_{\rm min}$ fast enough that the integral(s) on the right hand side of eq. (2.10) are well defined and vanish as $t \to T_{\rm min}$ and the solution of eq. (2.8) by

iterations makes sence♣ In this case we also see that $\zeta_n^{\mathrm{in}+}(\mathbf{x},t)$ are positive and $\zeta_n^{\mathrm{in}-}(\mathbf{x},t)$ are negative frequency solutions as $t \to T_{\mathrm{min}}$ which will allow a particle interpretation.

The general c-number solution to the equation of motion (2.1) is an arbitrary linear combination of the elements of the complete in-set. The Fermi-operator $\hat{\zeta}(\mathbf{x},t)$ is obtained from this by declaring the coefficients in front of the negative and positive frequency components to be the creation and annihilation operators respectively,

$$\hat{\zeta}(\mathbf{x},t) = \sum_{n=1}^{\infty} \left[\hat{a}_n^{\mathrm{in}} \zeta_n^{\mathrm{in}+}(\mathbf{x},t) + \hat{b}_n^{\mathrm{in}\dagger} \zeta_n^{\mathrm{in}-}(\mathbf{x},t) \right] \ . \tag{2.11}$$

Here $\hat{a}_n^{\mathrm{in}}$ is the annihilation operator of a particle with the energy E_n in the in-state, while $\hat{b}_n^{\mathrm{in}\dagger}$ is the creation operator of an anti-particle with the energy E_n in the in-state. Since the integrals on the right hand side of equation (2.10) vanish as $t \to T_{\mathrm{min}}$, these creation and annihilation operators obey the usual (free) anti-commutation relations and the in-vacuum state, $|0^{\mathrm{in}}\rangle$, is defined as:

$$\hat{a}_n^{\mathrm{in}}|0^{\mathrm{in}}\rangle = \hat{b}_n^{\mathrm{in}}|0^{\mathrm{in}}\rangle = 0 \ . \tag{2.12}$$

Gauge transforming eq. (2.11) back to the original notations,

$$\hat{\psi}(\mathbf{x},t) = \sum_{n=1}^{\infty} \left[\hat{a}_n^{\mathrm{in}} U_{\mathrm{in}}(\mathbf{x}) \zeta_n^{\mathrm{in}+}(\mathbf{x},t) + \hat{b}_n^{\mathrm{in}\dagger} U_{\mathrm{in}}(\mathbf{x}) \zeta_n^{\mathrm{in}-}(\mathbf{x},t) \right] \ , \tag{2.13}$$

we obtain the Fermi-operator in the in-representation.

Our next goal is to obtain a representation of $\hat{\psi}(\mathbf{x},t)$ in terms of the *out-* creation and annihilation operators. To do this we return to eq. (2.3) and repeat the previous steps with certain modifications. Consider a gauge transformation with the gauge function $U = U_{\mathrm{out}}(\mathbf{x})$ of eq. (2.2b),

$$A_\mu(\mathbf{x},t) = U_{\mathrm{out}}(\mathbf{x}) \big[\frac{i}{g} \partial_\mu + B_\mu^{\mathrm{out}}(\mathbf{x},t) \big] U_{\mathrm{out}}^\dagger(\mathbf{x}) \ , \tag{2.14a}$$

♣ This point was investigated in Ref. [10]. What is rather important for our applications in the next Section is the fact that the classical gauge field solutions of Ref. [7] *cannot* be cast in the form to allow iterations of the Yang-Feldman equation contrary to the claim of Ref. [10]. We will return to this point in Section 3. Here we just note that in order to apply the formalism of this Section to the case of classical fields in the background, the background should be modified at the early past and the far future to switch off the interactions with fermions.

$$\psi(\mathbf{x},t) = U_{\rm out}(\mathbf{x})\xi(\mathbf{x},t)\ . \tag{2.14b}$$

Equation (2.3) takes the form:

$$i\gamma^\mu \left(\partial_\mu - igB_\mu^{\rm out}(\mathbf{x},t)\right)\xi(\mathbf{x},t) = 0\ . \tag{2.15}$$

The background gauge field now is $B_\mu^{\rm out}$ which has to vanish in the far future. In this gauge fermions become free as $t \to T_{\rm max}$.

A complete orthonormal out-set of solutions of eq. (2.15), $\left\{\xi_n^{{\rm out}\pm}(\mathbf{x},t)\right\}_{n=1}^{\infty}$, is defined by the "initial" condition,

$$\xi_n^{{\rm out}\pm}(\mathbf{x},t) \to \psi_n^{\pm}(\mathbf{x})\, e^{\mp iE_n t} \quad {\rm as} \quad t \to T_{\rm max}\quad . \tag{2.16}$$

We now use the advanced Yang-Feldman form of the equation (2.15):

$$\xi_n^{{\rm out}\pm}(\mathbf{x},t) = \psi_n^{\pm}(\mathbf{x})\, e^{\mp iE_n t}\ -\ g\int_t^{T_{\rm max}} dy_0 \int d\mathbf{y}\Delta^{\rm adv}(x-y)\ \gamma^\mu B_\mu^{\rm out}(\mathbf{y},y_0)\ \xi_n^{{\rm out}\pm}(\mathbf{y},y_0)\ , \tag{2.17}$$

where $\Delta^{\rm adv}(x-y)$ is the advanced Green function,

$$i\gamma^\mu\partial_\mu\ \Delta^{\rm adv}(x-y) = \delta^{(4)}(x-y)\ , \tag{2.18a}$$

$$\Delta^{\rm adv}(x-y) \sim \theta(y_0 - x_0)\ . \tag{2.18b}$$

Now the (advanced) initial condition (2.16) is satisfied only for such backgrounds $B_\mu^{\rm out}$ that the integral on the right hand side of eq. (2.17) vanishes as $t \to T_{\rm max}$. The out-set elements are given by the iterative solution of equation (2.17):

$$\xi_n^{{\rm out}\pm}(\mathbf{x},t) = \psi_n^{\pm}(\mathbf{x})\, e^{\mp iE_n t}$$

$$- g\int_t^{T_{\rm max}} dy_0 \int d\mathbf{y}\Delta^{\rm adv}(x-y)\ \gamma^\mu B_\mu^{\rm out}(\mathbf{y},y_0)\ \psi_n^{\pm}(\mathbf{y})\, e^{\mp iE_n y_0}\ +\ \ldots\ . \tag{2.19}$$

$B_\mu^{\rm out}$ is required to vanish as $t \to T_{\rm max}$ that the integral(s) on the right hand side of eq. (2.19) are well defined and vanish$^\clubsuit$ as $t \to T_{\rm max}$.

The Fermi-operator $\hat{\xi}(\mathbf{x},t)$ is

$$\hat{\xi}(\mathbf{x},t) = \sum_{n=1}^{\infty}\left[\hat{a}_n^{\rm out}\xi_n^{{\rm out}+}(\mathbf{x},t) + \hat{b}_n^{{\rm out}\dagger}\xi_n^{{\rm out}-}(\mathbf{x},t)\right]\ , \tag{2.20}$$

with $\hat{a}_n^{\rm out}$ being the annihilation operator of a fermion and $\hat{b}_n^{\rm out\dagger}$ being the creation operator of an anti-fermion in the out-state. Since the integrals on the right hand side of equation (2.20) vanish as $t \to T_{\rm max}$, these out- creation and annihilation operators obey the usual (free) anti-commutation relations and the out-vacuum state, $|0^{\rm out}\rangle$, is:

$$\hat{a}_n^{\rm out}|0^{\rm out}\rangle = \hat{b}_n^{\rm out}|0^{\rm out}\rangle = 0 \ . \tag{2.21}$$

Gauge transforming eq. (2.20) back we obtain the Fermi-operator in the in-representation,

$$\hat{\psi}(\mathbf{x},t) = \sum_{n=1}^{\infty} \left[\hat{a}_n^{\rm out} U_{\rm out}(\mathbf{x}) \xi_n^{\rm out+}(\mathbf{x},t) + \hat{b}_n^{\rm out\dagger} U_{\rm out}(\mathbf{x}) \xi_n^{\rm out-}(\mathbf{x},t)\right] \ . \tag{2.22}$$

Equations (2.13) and (2.22) give two different representations of $\hat{\psi}(\mathbf{x},t)$ in terms of two complete sets, $\left\{\zeta_n^{\rm in\pm}(\mathbf{x},t)\right\}_{n=1}^{\infty}$ and $\left\{\xi_n^{\rm out\pm}(\mathbf{x},t)\right\}_{n=1}^{\infty}$, given by equations (2.10) and (2.19).

We now construct the operator of the fermionic current-density, $\hat{j}^\mu(x) = \hat{\bar{\psi}}(x)\gamma^\mu\hat{\psi}(x)$. We remind that $\hat{\psi}(x)$ is the left-handed fermion, so $\hat{j}^\mu(x)$ is a combination of an axial-vector and a vector current-density. We will require the vector charge to be conserved in the quantized theory (the theory remains gauge invariant) and the axial-vector charge will be violated anomalously.

The current-density operator, $\hat{j}^\mu(x)$, is a composite operator built out of local operators at the same space-time point x. For the integrals of the current-density, such as the charge operator, $\int d^3x \hat{j}^0(x)$, to be regular, the composite operator $\hat{j}^\mu(x)$ should be renormalized. The regularization should preserve gauge invariance. We use the ϵ-splitting regularization of Schwinger and define the renormalized current-density as

$$\hat{j}^\mu(x) = \lim_{\epsilon\to 0}\left(\hat{j}^\mu(x|\epsilon) - \{\text{counter term}\}^\mu\right) \ , \tag{2.23}$$

where the gauge invariant point-split current is

$$\hat{j}^\mu(x|\epsilon) = \hat{\bar{\psi}}(x+\epsilon/2)\gamma^\mu \ \mathcal{P}\exp\left[ig\int_{x-\epsilon/2}^{x+\epsilon/2} dy^\nu A_\nu(y)\right] \ \hat{\psi}(x-\epsilon/2) \ , \tag{2.24}$$

and the counter term is independent of the gauge field $A_\mu(x)$. The $\int d^3x\{\text{counter term}\}^0$ is a time-independent infinite constant to be subtracted from the unrenormalized charge operator to make the charge of the vacuum finite. Since the counter term is time-independent,

the effects of finite renormalization will cancel out in the difference of the charges in the beginning and at the end of the day.

To give $\hat{j}^\mu(x|\epsilon)$ the correct properties under Lorentz transformations, the limit $\epsilon \to 0$ should be taken symmetrically [11]:

$$\epsilon^\mu \to 0 \qquad \epsilon^\mu \epsilon^\nu / \epsilon^2 \to g^{\mu\nu}/4 \ . \tag{2.25}$$

Symmetric limit means that we first average over directions of ϵ and then let $\epsilon^2 = \epsilon^\mu \epsilon_\mu \to 0$.

Using the in-representation of the Fermi-operator, eq. (2.13), the ϵ-split current-density we find

$$\hat{j}^\mu(x|\epsilon) = \sum_{n=1}^{\infty} \left[\hat{a}_n^{\text{in}\dagger} \overline{\zeta^{\text{in}+}}_n U_{\text{in}}^\dagger + \hat{b}_n^{\text{in}} \overline{\zeta^{\text{in}-}}_n U_{\text{in}}^\dagger \right] |_{(x+\epsilon/2)} \cdot$$

$$\gamma^\mu \ \mathcal{P} \exp\big[ig \int_{x-\epsilon/2}^{x+\epsilon/2} dy^\nu A_\nu(y) \big] \ \sum_{m=1}^{\infty} \left[\hat{a}_m^{\text{in}} U_{\text{in}} \zeta_m^{\text{in}+} + \hat{b}_m^{\text{in}\dagger} U_{\text{in}} \zeta_m^{\text{in}-} \right] |_{(x-\epsilon/2)}$$

$$=: \hat{j}^\mu(x|\epsilon) :_{\text{in}} + S_{\text{in}}^{\epsilon\ \mu}[A] \ , \tag{2.26}$$

where $: \hat{j}^\mu(x|\epsilon) :_{\text{in}}$ is the normal form of $: \hat{j}^\mu(x|\epsilon) :_{\text{in}}$ with respect to the in- creation and annihilation operators and

$$S_{\text{in}}^{\epsilon\ \mu}[A] = \sum_{n=1}^{\infty} \overline{\zeta_n^{\text{in}-}}(x+\epsilon/2) \gamma^\mu \ \mathcal{P} \exp\big[ig \int_{x-\epsilon/2}^{x+\epsilon/2} dy^\nu B_\nu^{\text{in}}(y) \big] \ \zeta_n^{\text{in}-}(x-\epsilon/2) \ . \tag{2.27}$$

Here we used the anti-commutation relations and the gauge invariance of the point-split construction. The charge build from the normal ordered current-density $: \hat{j}^\mu(x|\epsilon) :_{\text{in}}$ is regular in the $\epsilon \to 0$ limit. Thus, the counter term can be chosen as follows:

$$\{\text{counter term}\}^\mu = S^{\epsilon\ \mu}[A \equiv 0] \equiv \sum_{n=1}^{\infty} \overline{\psi_n^-}(\mathbf{x} + \boldsymbol{\epsilon}/2) \gamma^\mu \psi_n^-(\mathbf{x} - \boldsymbol{\epsilon}/2) \ , \tag{2.28}$$

where $\psi_n^-(\mathbf{x})$ are negative energy eigenfunctions of the free Dirac Hamiltonian, eq. (2.7). (As it should be, the counter term is time-independent and does not depend on A_μ.)

The operator,

$$\hat{N}_{\text{i}} = \lim_{t \to T_{\min}} \int d^3x : \hat{j}^0(x|0) :_{\text{in}} = \sum_{n=1}^{\infty} \big(\hat{a}_n^{\text{in}\dagger} \hat{a}_n^{\text{in}} \ - \ \hat{b}_n^{\text{in}\dagger} \hat{b}_n^{\text{in}} \big) \ , \tag{2.29}$$

measures the net fermion number in the early past.

Similarly, the fermion number in the far future is given by

$$\hat{N}_{\rm f} = \lim_{t\to T_{\max}} \int d^3x : \hat{j}^0(x|0) :_{\rm out} = \sum_{n=1}^{\infty} \big(\hat{a}_n^{\rm out\dagger} \hat{a}_n^{\rm out} \ - \ \hat{b}_n^{\rm out\dagger} \hat{b}_n^{\rm out} \big) \ , \tag{2.30}$$

where $: \hat{j}^\mu(x|\epsilon) :_{\rm out}$ is the normal ordered current-density operator with respect to the out-creation and annihilation operators and,

$$\hat{j}^\mu(x|\epsilon) =: \hat{j}^\mu(x|\epsilon) :_{\rm out} + S_{\rm out}^{\epsilon\ \mu}[A] \ , \tag{2.31}$$

where,

$$S_{\rm out}^{\epsilon\ \mu}[A] = \sum_{n=1}^{\infty} \overline{\xi_n^{\rm out-}}(x+\epsilon/2)\ \gamma^\mu\ \mathcal{P}\exp\big[ig \int_{x-\epsilon/2}^{x+\epsilon/2} dy^\nu B_\nu^{\rm out}(y)\big]\ \xi_n^{\rm out-}(x-\epsilon/2) \ . \tag{2.32}$$

The fermion number violation is the expectation value of

$$\begin{aligned}\hat{N}_{\rm f} - \hat{N}_{\rm i} &= \lim_{t\to T_{\max}} \int d^3x : \hat{j}^0(x|0) :_{\rm out} \ - \lim_{t\to T_{\min}} \int d^3x : \hat{j}^0(x|0) :_{\rm in} \\ &= \int_{T_{\min}}^{T_{\max}} dt \int d^3x \ \lim_{\epsilon\to 0} \partial_t \hat{j}^0(x|\epsilon) \qquad (2.33) \\ - \lim_{\epsilon\to 0}\Big[\lim_{t\to T_{\max}} &\int d^3x \big(S_{\rm out}^{\epsilon\ 0}[A] - S^{\epsilon\ 0}[0]\big) \ - \ \lim_{t\to T_{\min}} \int d^3x \big(S_{\rm in}^{\epsilon\ 0}[A] - S^{\epsilon\ 0}[0]\big)\Big] \ .\end{aligned}$$

In deriving eq. (2.33) we used the fact that the counter term is a time-independent constant. The first term on the right hand side of eq. (2.33) can be written as,

$$\int_{T_{\min}}^{T_{\max}} dt \int d^3x \ \lim_{\epsilon\to 0} \partial_t \hat{j}^0(x|\epsilon) = \int d^4x \ \lim_{\epsilon\to 0} \partial_\mu \hat{j}^\mu(x|\epsilon) \ , \tag{2.34}$$

since the boundary terms at the surface at the spatial infinity (at finite time, $T_{\min} < t < T_{\max}$) are vanishing. As a result of a direct computation [11] we also have,

$$\int d^4x \ \lim_{\epsilon\to 0} \partial_\mu \hat{j}^\mu(x|\epsilon) = \frac{g^2}{16\pi^2} \frac{1}{2} \epsilon^{\mu\nu\alpha\beta} \int d^4x {\rm tr}(F_{\mu\nu} F_{\alpha\beta}) \equiv Q \ . \tag{2.35}$$

The expression above is obtained by differentiating the right hand side of eq. (2.24), making use of Dirac equation (2.3) and finally taking the symmetric limit $\epsilon \to 0$ as prescribed by eq. (2.25). This way of obtaining the expression on the right hand side of eq. (2.34) can be viewed as a derivation of the anomaly equation (1.2).

The second term on the right hand side of eq. (2.33) can be abbreviated as $-(q^{\rm out} - q^{\rm in})$. Here $q^{\rm out}$ and $q^{\rm in}$ are the "fermion" charges of the radiating gauge fields $B_\mu^{\rm out}$ and $B_\mu^{\rm in}$ and have nothing to do with the actual number of fermions. They can be calculated by substituting iterative solutions$^\heartsuit$ of the Yang-Feldman equations (2.17) and (2.8) into the

$^\heartsuit$ Equations (2.17) and (2.18) should be iterated three times

expressions for S, eq. (2.32), (2.27) and first performing the integrations over the three-space in (2.33) and only then letting t to go to the infinite future or infinite past. The other order of limits would be inconsistent with our set up (and would give zero result). $q^{\rm out}$ and $q^{\rm in}$ were calculated by N. Christ [9],

$$q^{\rm out} = \lim_{t\to+\infty} \int d^3x K^0[B^{\rm out}] \ , \tag{2.36a}$$

$$q^{\rm in} = \lim_{t\to-\infty} \int d^3x K^0[B^{\rm in}] \ , \tag{2.36b}$$

where $K_0[A]$ is a zeroth component of the topological current,

$$K^\mu[A] = \frac{g^2}{16\pi^2} \int d^3x \epsilon^{\mu\nu\alpha\beta} \ {\rm tr}\big(A_\nu F_{\alpha\beta} - \frac{2}{3} A_\nu A_\alpha A_\beta\big) \ , \tag{2.37}$$

and

$$\partial_\mu K^\mu = \frac{g^2}{16\pi^2} \frac{1}{2} \epsilon^{\mu\nu\alpha\beta} {\rm tr}(F_{\mu\nu} F_{\alpha\beta}) \ . \tag{2.38}$$

We note that $q^{\rm out}$ and $q^{\rm in}$ are gauge invariant under small gauge transformations while large gauge transformations would be inconsistent with our requirements on $B^{\rm out}$ and $B^{\rm in}$ of falling off with time and should be absorbed into $U_{\rm out}$ and $U_{\rm in}$.

Putting all the bits together, we reproduce N. Christ's result [9]:

$$\langle \hat{N}_{\rm f} - \hat{N}_{\rm i} \rangle = Q - q^{\rm out} + q^{\rm in} \ . \tag{2.39}$$

Thus, when there is a radiation field, $B_\mu^{\rm (in)out}$, present in the initial or final state, the net violation of the classically conserved number of chiral fermions is not given by the integral of the axial-vector anomaly (topological charge Q), but additional subtractions must be made [9]. The so-called fermionic charge, $\int d^3x \hat{\jmath}^0$, contains a piece $q^{\rm (in)out}$ which is the "fermion" charge* of the radiating gauge field $B_\mu^{\rm (in)out}$ and has not much to do with the actual number of fermions which in its turn is measured by a corresponding normal ordered product.

---

* The $B_\mu^{\rm (in)out}$ fields should go to zero as $t \to T_{\rm (min)max}$ in order to have free fermions at early and late times and iterate Yang-Feldman equations, but this does not guarantee that $q^{\rm (in)out}$ necessarily vanish due to the order of limits in eqs. (2.36)

This is rather interesting since the topological charge Q does not have to be an integer [7] and one may hope that the subtraction of $q^{(\mathrm{in})\mathrm{out}}$ will somehow make the net effect of the fermion number violation to be an integer$^\sharp$,

We will show now that the Christ's result, eq. (2.39), can be put in the form in which the fermion number is always violated by an *integer* amount for arbitrary gauge field in the background which allows iterations of the Yang-Feldman equations (2.17), (2.8). We have,

$$\langle \hat{N}_{\mathrm{f}} - \hat{N}_{\mathrm{i}} \rangle = Q - q^{\mathrm{out}} + q^{\mathrm{in}}$$

$$= \lim_{T\to+\infty} \int_{-T}^{T} dt \int d^3x\, \partial_\mu K^\mu[A] - \lim_{t\to+\infty} \int d^3x K^0[B^{\mathrm{out}}] + \lim_{t\to-\infty} \int d^3x K^0[B^{\mathrm{in}}]$$

$$= \int d^3x\ K^0\big[U_{\mathrm{out}}(\mathbf{x}) \frac{i}{g} \partial_\mu U^\dagger_{\mathrm{out}}(\mathbf{x})\big] \ - \ \int d^3x\ K^0\big[U_{\mathrm{in}}(\mathbf{x}) \frac{i}{g} \partial_\mu U^\dagger_{\mathrm{in}}(\mathbf{x})\big]$$

$$\equiv \nu[U_{\mathrm{out}}] - \nu[U_{\mathrm{in}}] \ \in \mathcal{Z} \ , \tag{2.40}$$

which is an integer since the winding numbers of $U_{(\mathrm{in})\mathrm{out}}$ are integer by construction.

An important thing is to make sure that the integer baryon number violation is not always zero for example on Minkowski space classical solutions. In the next Section we will calculate the fermion number violation in the background of the spherical solutions [7]. We will demonstrate that it is integer and non-zero in general and also derive some useful selection rules.

3. Classical Solutions in the Spherical Ansatz and Fermion Number Violation

Working in the spherical ansatz for pure $SU(2)$ gauge theory we will first review how [7] the equations of motion can be reduced to two equations for two gauge invariant variables ρ^2 and ψ. Then we will discuss classical solutions in (3+1)-dimensional Minkowski space and calculate the violation of the fermion number in their background.

The action for pure $SU(2)$ gauge theory is

$$S = -\frac{1}{2} \int d^4x \ \mathrm{tr}\,(F_{\mu\nu} F^{\mu\nu})\,, \tag{3.1}$$

$^\sharp$ It would have been rather unpleasant to find a non-integer number of fundamental fermions in the detector at the end of a scattering experiment

where $F_{\mu\nu} = F^a_{\mu\nu}\,(\sigma^a/2) = \partial_\mu A_\nu - \partial_\nu A_\mu - ig\,[A_\mu, A_\nu]$ is the field strength and $A_\mu = A^a_\mu\,(\sigma^a/2)$.

The spherical ansatz [12] is given in terms of the four functions a_0, a_1, α, β by

$$
\begin{aligned}
A_0(\mathbf{x},t) &= \frac{1}{2g}\; a_0(r,t)\; \boldsymbol{\sigma}\cdot\hat{\mathbf{x}} \quad , \\
A_i(\mathbf{x},t) &= \frac{1}{2g}\; \left(a_1(r,t)\; \boldsymbol{\sigma}\cdot\hat{\mathbf{x}}\hat{x}_i + \frac{\alpha(r,t)}{r}\;(\sigma_i - \boldsymbol{\sigma}\cdot\hat{\mathbf{x}}\hat{x}_i) + \frac{1+\beta(r,t)}{r}\epsilon_{ijk}\hat{x}_j\sigma_k \right) \quad ,
\end{aligned}
\tag{3.2}
$$

where $\hat{\mathbf{x}}$ is a unit three-vector in the radial direction. Note that $1/g$ factors are introduced in eqs. (3.2) as was done in Refs. [12,7] which makes equations of motion g-independent. This was not so in the treatment of Ref. [8]. Perturbative solutions of Ref. [8] will be mentioned in the next Section.

The action (3.1) in the spherical ansatz takes the form

$$
S = \frac{4\pi}{g^2} \int dt \int_0^\infty dr \left(-\frac{1}{4} r^2 f_{\mu\nu} f^{\mu\nu} - (D_\mu\chi)^* \, D^\mu\chi - \frac{1}{2r^2}\left(|\chi|^2 - 1\right)^2 \right) \quad . \tag{3.3}
$$

where $f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu$ with $\mu, \nu = t, r$, is the (1+1)-dimensional field strength, $\chi = \alpha + i\beta$ is a complex scalar and $D_\mu\chi = (\partial_\mu - i a_\mu)\,\chi$ is the covariant derivative. To keep up with notations of Ref. [7], in the spherical ansatz indices are raised and lowered with the $1+1$ dimensional metric $\eta_{\mu\nu} = \mathrm{diag}(-1,+1)$.

The ansatz (3.2) preserves a residual $U(1)$ subgroup of the $SU(2)$ gauge group consisting of the transformations,

$$
U(\mathbf{x},t) = \exp\left[i\Omega(r,t) \frac{\boldsymbol{\sigma}\cdot\hat{\mathbf{x}}}{2} \right] \quad . \tag{3.4}
$$

These induce the gauge transformations

$$
a_\mu \to a_\mu + \partial_\mu\Omega \quad , \qquad \chi \to \exp(i\Omega)\chi \quad , \tag{3.5}
$$

which leave (3.3) invariant.

The $(1+1)$-dimensional equations of motion for the reduced theory (3.4) are given by

$$
-\partial^\mu \left(r^2 f_{\mu\nu} \right) = i\left[(D_\nu\chi)^* \, \chi - \chi^* D_\nu\chi \right] \quad , \tag{3.6a}
$$

$$
\left(-D^2 + \frac{1}{r^2}\left(|\chi|^2 - 1\right) \right) \chi = 0 \ . \tag{3.6b}
$$

Let us express the complex scalar field χ in polar form,

$$\chi(r,t) = -i\rho(r,t)\exp\left[i\varphi(r,t)\right] \quad , \tag{3.7}$$

where ρ and φ are real scalar fields and $\rho(r,t) \geq 0$.

One must bear in mind that in a point ρ where vanishes the angle φ is not defined. Assume that ρ vanishes at a single point (r_*, t_*). Surround the point (r_*, t_*) by a simple closed contour in the (r,t)-space. Then, since χ is continuous and $\rho \neq 0$ on the contour, the change of φ along the contour is in general an integer multiple of 2π. This integer multiple will be called a *degree* of φ in the point (r_*, t_*). Degree of φ is non-zero only if φ changes discontinuously in the point (r_*, t_*) which is called then a singular point.

One of the central results of this Section will be a derivation of the selection rule: *the change of the numbers of fermions is equal to the sum of the degrees of φ in each singular point*. This is an integer by construction (which cannot [7] be said about the topological charge).

In terms of ρ, φ and a_μ, the four equations contained in (3.6) read

$$\partial^\mu\left(r^2 f_{\mu\nu}\right) + 2\rho^2\left(\partial_\nu\varphi - a_\nu\right) = 0 \quad , \tag{3.8a}$$

$$\partial^\mu\partial_\mu\rho - \rho\left(\partial^\mu\varphi - a^\mu\right)\left(\partial_\mu\varphi - a_\mu\right) - \frac{1}{r^2}\rho\left(\rho^2 - 1\right) = 0 \quad , \tag{3.8b}$$

and

$$\partial^\mu\left[\rho^2(\partial_\mu\varphi - a_\mu)\right] = 0 \ . \tag{3.8c}$$

The last equation follows from (3.8a) so there are three, not four, independent equations, as expected because of the residual $U(1)$ gauge invariance.

In practice the new field $\rho \equiv \sqrt{\alpha^2 + \beta^2}$ is not very convenient since it involves the square root of the old variables. It will be more useful for us to use $\rho^2 = \alpha^2 + \beta^2$ as the new primary field variable instead of ρ. By rewriting eq. (3.8b) as

$$\frac{1}{2}\partial^\mu\partial_\mu\rho^2 - \frac{1}{4\rho^2}(\partial^\mu\rho^2)(\partial_\mu\rho^2) - \rho^2\left(\partial^\mu\varphi - a^\mu\right)\left(\partial_\mu\varphi - a_\mu\right) - \frac{\rho^2}{r^2}\left(\rho^2 - 1\right) = 0 \quad , \tag{3.8b'}$$

we ensure that only ρ^2 and not ρ appears in the classical equations.

Since in (1+1) dimensions $f_{\mu\nu}$ must be proportional to $\epsilon_{\mu\nu}$, we define [7] a new field ψ as follows:

$$r^2 f_{\mu\nu} = -2\epsilon_{\mu\nu}\psi \quad , \tag{3.9}$$

here $\epsilon_{01} = +1$. Equation (3.8a) now becomes

$$\partial^\alpha \psi = -\epsilon^{\alpha\nu} \rho^2 (\partial_\nu \varphi - a_\nu) \quad . \tag{3.10}$$

which implies

$$\partial_\alpha \left(\frac{\partial^\alpha \psi}{\rho^2} \right) - \frac{2}{r^2} \psi = 0 \ . \tag{3.11}$$

This gives an equation solely in terms of the fields ρ^2 and ψ. We may also use (3.10) to express the second term in (3.8b') in terms of only ρ^2 and ψ,

$$-\partial_t^2 \rho^2 + \partial_r^2 \rho^2 + \frac{1}{2\rho^2} \left((\partial_t \rho^2)^2 - (\partial_r \rho^2)^2 \right) - \frac{2}{\rho^2} \left((\partial_t \psi)^2 - (\partial_r \psi)^2 \right) - \frac{2\rho^2}{r^2} (\rho^2 - 1) = 0 \ , \tag{3.12a}$$

$$-\partial_t \Big(\frac{\partial_t \psi}{\rho^2} \Big) + \partial_r \Big(\frac{\partial_r \psi}{\rho^2} \Big) - \frac{2\psi}{r^2} = 0 \ . \tag{3.12b}$$

Equations (3.12) are equivalent to the original eqs. (3.6), but now the fields are ρ^2 and ψ which are gauge invariant, and there are only two equations in (3.12).

Using the equations of motion, the energy associated with the action (3.3) can be written in terms of ρ^2 and ψ as

$$\begin{aligned} E = \frac{8\pi}{g^2} \int_0^\infty dr \Bigg[& \frac{1}{8\rho^2} \left(\partial_t \rho^2 \right)^2 + \frac{1}{8\rho^2} \left(\partial_r \rho^2 \right)^2 + \frac{1}{2\rho^2} \left(\partial_t \psi \right)^2 \\ & + \frac{1}{2\rho^2} \left(\partial_r \psi \right)^2 + \frac{\psi^2}{r^2} + \frac{\left(\rho^2 - 1 \right)^2}{4r^2} \Bigg] \ . \end{aligned} \tag{3.13}$$

We are interested in finite energy solutions to (3.12).

Witten [12] observed that (3.3) is the action for an Abelian Higgs model in a curved space-time. In fact [7], the space-time manifold is the two dimensional De Sitter space, i.e. hyperboloid $z_0^2 - z_1^2 - z_2^2 = -1$ where the z_i are functions of r and t and the coordinates r and t cover only half of the hyperboloid for which $z_0 + z_2 > 0$. It is rather convenient to work with coordinates w and τ that live on the hyperboloid. The coordinate w is a bounded measure of the vertical position along the hyperboloid, $|w| < \pi/2$, and τ measures the azimuthal angle, $|\tau| \le \pi$. For more details see Fig. 1 of Ref. [7]. The explicit representation of w and τ is given by

$$w = \ \arctan\Big(\frac{1 + t^2 - r^2}{2r} \Big) \ , \tag{3.14a}$$

$$\tau = \ \mathrm{sign}(\tau) \ \arccos\Big(\frac{1 - t^2 + r^2}{\sqrt{(1 + t^2 - r^2)^2 + 4r^2}} \Big) \ . \tag{3.14b}$$

In terms of w-τ variables equations of motion (3.12) take the form [7]

$$-\partial_\tau^2\rho+\partial_w^2\rho+\frac{\left(\partial_\tau\rho^2\right)^2}{2\rho^2}-\frac{\left(\partial_w\rho^2\right)^2}{2\rho^2}-\frac{2\left(\partial_\tau\psi\right)^2}{\rho^2}+\frac{2\left(\partial_w\psi\right)^2}{\rho^2}-\frac{2\rho^2\left(\rho^2-1\right)}{\cos^2 w}=0\ , \tag{3.15a}$$

$$-\partial_\tau\left(\frac{\partial_\tau\psi}{\rho^2}\right)+\partial_w\left(\frac{\partial_w\psi}{\rho^2}\right)-\frac{2\psi}{\cos^2 w}=0\ . \tag{3.15b}$$

As a characteristic example of finite energy solutions to equations of motion (3.15) we consider solutions of Lüscher and Schechter [13]. These solutions have finite energy, finite action and non-trivial topological charge [7]. As was shown in Ref. [8], these explicit solutions are examples of a wide class of finite energy solutions all of which have certain general features in common. At early times they depict a thin spherical shell of energy imploding towards the origin at near the speed of light. At around zero time the region around the origin is energetically excited and at late times the shell is expanding outward, asymptotically approaching the speed of light.

The main advantage of Lüscher - Schechter solutions is that they are known analytically:

$$\rho^2(w,\tau)=1+q(\tau)\ \big(q(\tau)+2\big)\cos^2 w\ , \tag{3.16a}$$

$$\psi(w,\tau)=\frac{1}{2}\ \frac{dq(\tau)}{d\tau}\ \cos^2 w\ , \tag{3.16b}$$

where the function $q(\tau)$ is a solution of the ordinary differential equation:

$$\ddot q+2q(q+1)(q+2)=0\ . \tag{3.17}$$

The mechanical problem associated with eq. (3.17) is that of a classical particle trapped in the double well potential $U=\frac{1}{2}q^2(\mathrm{q}+2)^2$. The "energy" ε of the "particle" is

$$\varepsilon=\frac{1}{2}\dot q^2+U(q)\ . \tag{3.18}$$

General solution of (3.17) will depend on the "energy" ε and the "time"-translation parameter τ_0. There are two classes of solutions depending on whether ε is smaller or larger than 1/2, the barrier height of $U(q)$ at the unstable point $q=-1$:

$$\begin{aligned} q(\tau)&=-1\pm(1+\sqrt{2\varepsilon})^{1/2}\mathrm{dn}\big((1+\sqrt{2\varepsilon})^{1/2}(\tau-\tau_0)\mid m_1\big)\\ m_1&=2\sqrt{2\varepsilon}/(1+\sqrt{2\varepsilon});\qquad \varepsilon\le 1/2\ \ , \end{aligned} \tag{3.19}$$

and

$$q(\tau) = -1 \pm (1+\sqrt{2\varepsilon})^{1/2}\mathrm{cn}\big((8\varepsilon)^{1/4}(\tau-\tau_0) \mid m_2\big)$$
$$m_2 = (1+\sqrt{2\varepsilon})/(2\sqrt{2\varepsilon}); \qquad \varepsilon > 1/2 \quad , \tag{3.20}$$

where $\mathrm{dn}(u|m)$ and $\mathrm{cn}(u|m)$ are the Jacobi elliptic functions‡

$$u = \int_{\mathrm{dn}(u|m)}^{1} \frac{dt}{\sqrt{(1-t^2)(t^2+m-1)}} \ , \tag{3.21a}$$

$$u = \int_{\mathrm{cn}(u|m)}^{1} \frac{dt}{\sqrt{(1-t^2)(mt^2-m+1)}} \ , \tag{3.21b}$$

There are always two forms of solutions ($\pm$ signs in (3.18) and (3.19)) since eq. (3.17) is not changed by the substitution $q = 1+\kappa \to 1-\kappa$. In particular when $\varepsilon < 1/2$, different signs in eq. (3.19) correspond to the particle being trapped in different wells. The parameter τ_0 corresponds to the time at which the particle moving in the potential $U(q)$ with energy ε is at a turning point.

The Lüscher-Schechter solutions can also be represented in terms of the four original functions of the spherical ansatz

$$\begin{aligned} a_\mu &= -q(\tau)\ \partial_\mu w \ , \\ \alpha &= \frac{1}{2} q(\tau)\ \sin 2w \ , \\ \beta &= -(1 + q(\tau)\cos^2 w) \ , \end{aligned} \tag{3.22}$$

where $\mu = t, r$.

The Lüscher-Schechter solutions give spherically symmetric waves of localized energy density. Now we would like to discuss the solution itself, i.e. $\rho^2(r,t)$ and $\psi(r,t)$. Figures 1 and 2 show the r-profiles of $\rho^2(r,t)$ and $\psi(r,t)$ given by eqs. (3.16) for a sequence of negative and positive times for a specific case of $\tau_0 = 1$ and $\varepsilon = 1$. In the distant past the "two-dimensional" fields $\rho^2(r,t)$ and $\psi(r,t)$ are the incoming wave packets in the r-space which propagate undistorted in a soliton-like manner at near the speed of light. At around zero time the packets distort, collapse and bounce back producing outgoing wave packets. At large enough positive time the outgoing wave packets again propagate undistorted approaching the speed of light. These $\rho^2(r,t)$- and $\psi(r,t)$- packets represent imploding or

‡ Since there are several incompatible conventions in common mathematical use we will always be using here notations of *Mathematica* [14]

expanding spherical shells in (3+1) dimensions. As the shell expands it leaves the region of space behind it in a pure gauge configuration. In the (1+1) dimensional (r,t)-space the outgoing wave packets move undistorted.

As we already pointed out, these are the properties of not just Lüscher - Schechter solutions, but of a wide class of spherically symmetric solutions [8]. Indeed, consider equations (3.12) and imagine that at some early time $t = T_{\rm i} \ll 0$ the fields $\delta \equiv \rho^2 - 1$ and ψ are both pulses of width Δ centered at r near $|T_{\rm i}|$ with $\Delta \ll |T_{\rm i}|$. By a pulse we mean here a function which is very close to zero except in a region of the size Δ. For $r \sim |T_{\rm i}| \gg \Delta$ we can now neglect the $1/r^2$ terms in eqs. (3.12). We then see that if $\psi(r,t)$ and $\delta(r,t) \equiv \rho^2(r,t) - 1$ depend only on $r+t$, that is $\psi(r,t) = \psi_{\rm p}(r+t)$ and $\delta(r,t) \equiv \rho^2(r,t) - 1 = \delta_{\rm p}(r+t)$ then eqs. (3.12) are satisfied. Since $\psi_{\rm p}(u)$ and $\delta_{\rm p}(u)$ are close to zero except for $u \sim \Delta$, the solution $\psi(r,t)$ and $\rho^2(r,t)$ describe incoming wave packets of the width Δ moving undistorted along $r = -t$. This description remains valid for all $t \ll -\Delta$.

At the late time $t = T_{\rm f} \gg 0$ the $1/r^2$ terms in eqs. (3.12) can be neglected again and the solution is described by pulses again, $\psi(r,t) = \tilde{\psi}_{\rm p}(r-t)$ and $\delta(r,t) \equiv \rho^2(r,t) - 1 = \tilde{\delta}_{\rm p}(r-t)$ where $\tilde{\psi}_{\rm p}(v)$ and $\tilde{\delta}_{\rm p}(v)$ are some new pulses of a width Δ and this is valid for all $t \gg \Delta$.

We now return to Fig. 1 since there is one more important lesson to be learned from Lüscher - Schechter solutions. It is apparent from Fig. 1a that there is a point in the (r,t)-space, (r_*, t_*), such that $\rho^2(r_*, t_*) = 0$. We will show now that φ does change discontinuously in the point (r_*,t_*) and the *degree* of φ in the point (r_*,t_*) is 1.

It follows from equation (3.16a) that the ρ^2-component of an arbitrary Lüscher - Schechter solution can vanish at the point $\tau_* = \tau(r_*, t_*)$, $w_* = w(r_*, t_*)$ in the (τ, w)-space if and only if:

$$q(\tau_*) = -1 \ , \tag{3.23a}$$

$$\cos^2 w_* = 1 \ . \tag{3.23b}$$

The first condition can be satisfied only if $\varepsilon \geq 1/2$ since $q = -1$ is the height of the barrier, $U(q=-1) = 1/2$. Thus, solutions of the class (3.19) have non-vanishing ρ^2 and a continuous φ at any r and t. These solutions describing a “particle” trapped in a well will not cause fermion number violation. We now turn to solutions of the class (3.20). The

condition (3.23a) implies,

$$\tau_{*n} = \tau_0 + \frac{1}{(8\varepsilon)^{1/4}} u_{*n} \ , \tag{3.24}$$

where $u = u_{*n}$ with $n = -\infty..\infty$ are the roots of $\mathrm{cn}(u_{*n}|m_2)$ given by

$$u_{*n} = \int_0^{\pi/2+\pi n} \frac{d\theta}{\sqrt{1 - m_2 \sin^2\theta}} = (2n+1)\ \mathrm{K}(m_2)\ . \tag{3.25}$$

Solving conditions (3.23b) and (3.25) with the help of eqs. (3.14), we have:

$$r_{*n} = \sqrt{1 + t_{*n}^2}\ , \tag{3.26a}$$

$$t_{*n} = \tan\left(\tau_0 + \frac{1+2n}{(8\varepsilon)^{1/4}}\ \mathrm{K}\big(\frac{1+\sqrt{2\varepsilon}}{2\sqrt{2\varepsilon}}\big)\right)\ , \tag{3.26b}$$

$$n\ :\quad -\frac{\pi}{2} \leq \tau_0 + \frac{1+2n}{(8\varepsilon)^{1/4}}\ \mathrm{K}\big(\frac{1+\sqrt{2\varepsilon}}{2\sqrt{2\varepsilon}}\big) \leq \frac{\pi}{2}\ . \tag{3.26c}$$

Equation (3.26c) ensures that there is a certain *finite* number of times ρ^2 vanishes, in particular, for the case of $\tau_0 = 1$ and $\varepsilon = 1$, there is only one n allowed by eq. (3.26c), which is $n = -1$. This gives a single point $(r_{-1*}, t_{-1*}) \simeq (1.099, -0.455)$, which is consistent with Fig. 1a.

In general, ρ^2 vanishes each time the "particle" of the mechanical system (3.17)-(3.18) goes over the top of the potential at $q = -1$ which we can call the "spahaleron of the double well". Since the "time" coordinate, τ, of the mechanical analog is not the time t of the real world, but has a compact support on the hyperboloid (3.14), the "particle" goes through the "sphaleron" only a *finite* number of times (determined by eq. (3.26c)), each time approaching it from the different side. We will see that each time this happens, the fermion number is violated by ± 1.

In fact with some algebra one can see that for an arbitrary Lüscher - Schechter solution φ changes discontinuously in each (r_{*n}, t_{*n}) of eqs. (3.26) and the *degree* of φ in each of these points is ± 1. This can be proven by expanding α and β around the (r_*, t_*)-point,

$$\alpha \sim -(r - r_*) + \frac{t_*}{r_*}(t - t_*)\ , \tag{3.27a}$$

$$\beta \sim \dot{q}(\tau_*)\frac{t_*}{r_*}(r - r_*) - \dot{q}(\tau_*)(t - t_*)\ , \tag{3.27b}$$

and evaluating the winding of the polar angle of $\alpha+i\beta$ along an infinitesimal circle around the (r_*, t_*)-point.

We are now ready to consider the fermion number violation in the presence of the classical solutions in the spherical ansatz. We assume that the classical equations (3.12) are solved and the fields $\rho^2(r,t)$ and $\psi(r,t)$ are known. In order to obtain the (3+1) dimensional form of the solution, $A_\mu(\mathbf{x},t)$ we have to find $a_0(r,t)$, $a_1(r,t)$, $\alpha(r,t)$ and $\beta(r,t)$ in terms of $\rho^2(r,t)$ and $\psi(r,t)$ in a given gauge. From eq. (3.10) we have

$$\partial_t\psi = (a_1 - \partial_r\varphi)\ \rho^2\ , \tag{3.28a}$$

$$\partial_r\psi = (a_0 - \partial_t\varphi)\ \rho^2\ . \tag{3.28b}$$

If ρ^2 was non-vanishing at any r and t we could make $\varphi(r,t) = 0$ at any r and t by a continuous gauge transformation. In this case we would have $a_0 = \partial_r\psi/\rho^2$ and $a_1 = \partial_t\psi/\rho^2$ at any r and t. Such continuous gauge transformation $\Omega(r,t) = -\varphi(r,t)$ does not exist if φ changes discontinuously when ρ^2 goes through zero.

Suppose now that there is a single singular point (r_*, t_*) where ρ^2 vanishes and φ changes discontinuously. For definitness we start with a gauge $a_0 = 0$. In this gauge we have from eq. (3.28b):

$$\varphi(r,t) = -\int_{\mathcal{C}_{(r,T_{\rm min})\mapsto(r,t)}} \frac{\partial_r\psi}{\rho^2}\ dt\ , \tag{3.29}$$

where we also put $\varphi(r, T_{\rm min}) = 0$ by exhausting the initial gauge freedom. The contour of integration $\mathcal{C}_{(r,T_{\rm min})\mapsto(r,t)}$ runs from $T_{\rm min}$ to t surrounding the singularity (r_*, t_*) on the left as shown on Fig. 3. The polar angle variable $\varphi(r,t)$ of eqn (3.29) is discontinuous on a ray $\{t = t_*, r \geq r_*\}$. We can still make a continuous gauge transformation (3.4) with some $\Omega_1(r,t)$ which will make $\varphi(r,t) = 0$ at $t \ll t_*$ and all r. For example we can chose

$$\Omega_1(r,t) = \int_{T_{\rm min}}^{t} \frac{\partial_r\psi(r,\tau)}{\rho^2(r,\tau) + h(\tau)} d\tau\ , \tag{3.30a}$$

with $h(\tau)$ being a positive function with a support only at $\tau \sim t_*$. Thus, h makes Ω_1 well defined at (r_*, t_*) and continuous, but can be dropped at all $t \ll t_*$. We will call this (specified by Ω_1) gauge an initial gauge.

On the other hand the gauge $a_0 = 0$, $\varphi(r, T_{\rm min}) = 0$ can be related by a different continuous gauge transformation $\Omega_2(r,t)$ with what will be called a final gauge in which

$\varphi(r,t)=0$ at $t \gg t_*$ and all r. We chose

$$\Omega_2(r,t) = -\int_t^{T_{\max}} \frac{\partial_r\psi(r,\tau)}{\rho^2(r,\tau)+h(\tau)} d\tau + \int_{\mathcal{C}_{(r,T_{\min})\mapsto(r,T_{\max})}} \frac{\partial_r\psi(r,\tau)}{\rho^2(r,\tau)} \, d\tau \ . \tag{3.30b}$$

Finally, we relate the initial gauge in which $\varphi(r,t)=0$ at $t \ll t_*$ with the final gauge in which $\varphi(r,t)=0$ at $t \gg t_*$ by the gauge transformation $\Omega_{\rm f}(r) = \Omega_2(r,t) - \Omega_1(r,t)$.

In the initial gauge the vector potential at early and late times is given by:

$$A_\mu(\mathbf{x}, t \ll t_*) = B_\mu(\mathbf{x},t) \ , \tag{3.31a}$$

$$A_\mu(\mathbf{x}, t \gg t_*) = U_{\rm f}(\mathbf{x})\big[\frac{i}{g}\partial_\mu + B_\mu(\mathbf{x})\big]U_{\rm f}^\dagger(\mathbf{x},t) \ . \tag{3.31b}$$

Here the field $B_\mu(\mathbf{x},t)$ is just the right hand side of eqs. (3.2) with

$$a_0(r,t) = \partial_r\psi(r,t)/\rho^2(r,t) \ , \tag{3.32a}$$

$$a_1(r,t) = \partial_t\psi(r,t)/\rho^2(r,t) \ , \tag{3.32b}$$

$$\alpha(r,t) = 0 \ , \tag{3.32c}$$

$$1+\beta(r,t) = 1-\rho(r,t) \ . \tag{3.32d}$$

The gauge transformation $U_{\rm f}(\mathbf{x},t)$ is an $SU(2)$-valued continuous function of $\mathbf{x}$,

$$U_{\rm f}(\mathbf{x}) = \exp\left[i\Omega_{\rm f}(r)\frac{\boldsymbol{\sigma}\cdot\hat{\mathbf{x}}}{2}\right] \ , \tag{3.33}$$

where

$$\Omega_{\rm f}(r) = -\int_{T_{\min}}^{T_{\max}} \frac{\partial_r\psi(r,\tau)}{\rho^2(r,\tau)+h(\tau)} d\tau + \int_{\mathcal{C}_{(r,T_{\min})\mapsto(r,T_{\max})}} \frac{\partial_r\psi}{\rho^2} d\tau \ . \tag{3.34}$$

First, we notice that $U_{\rm f}(\mathbf{x})$ is, in fact, t-independent since the contours of integration on the right hand side of eq. (3.34) are t-independent. We also note that, since the ψ wave packets are localized in the vicinity of the light-cone, $\Omega_{\rm f}(r) = 2\pi \cdot \textit{degree}(\varphi(r_*,t_*))$ and $U_{\rm f}(\mathbf{x}) = 1$ for $|\mathbf{x}| \gg \max(T_{\max}, |T_{\min}|)$. Thus, $U_{\rm f}(\mathbf{x})$ defines a mapping of a three-sphere into a three-sphere which can be characterized by an integer winding number $\nu(U_{\rm f})$ which is equal to $\textit{degree}(\varphi(r_*,t_*))$.

A practical example for the discussion above is a special case of a Lüscher - Schechter solution with $\tau_0 = 1$ and $\varepsilon = 1$, which has only one singular point $(r_{-1*}, t_{-1*}) \simeq$

$(1.099, -0.455)$, given by eqs. (3.26). Degree of $\varphi(r,t)$ in this point is 1 (and it can also be checked explicitly that the right hand side of eq. (3.29) changes discontinuously by 2π in this point).

Now we have to consider the violation of the fermion number in the presence of a background gauge field of eqs. (3.31)-(3.34). Using the formalism of Section 2, fermion number violation can be calculated in the presence of the background of the type (2.2). The ansatz (2.2) can be reduced to the form (3.31) with $U_{\rm in}(\mathbf{x}) = 1$, $U_{\rm out}(\mathbf{x}) = U_{\rm f}(\mathbf{x})$ and $B_\mu^{\rm (in)out}(\mathbf{x},t) = B_\mu(\mathbf{x},t)$. We have seen already that the gauge transformations $U_{\rm in}(\mathbf{x}) = 1$ and $U_{\rm out}(\mathbf{x}) = U_{\rm f}(\mathbf{x})$ are continuous functions of $\mathbf{x}$ which, as $\mathbf{x} \to \infty$, approach direction-independent constants which satisfies the requirements of Section 2. But the gauge fields $B_\mu^{\rm in}(\mathbf{x},t)$ and $B_\mu^{\rm out}(\mathbf{x},t)$ on the right hand side of eqs. (2.2) were required to have essentially *finite* support in the $\mathbf{x}$-space at any fixed time t and to vanish at any $\mathbf{x}$ as time goes respectively to $T_{\rm min}$ or $T_{\rm max}$. The first requirement of the essentially *finite* support in the $\mathbf{x}$-space is easily satisfied which follows from eqs. (3.2), (3.32) and the fact that for classical solutions ψ and $1-\rho$ are well localized pulses at early and late times. On the other hand, the second requirement that the $B_\mu(\mathbf{x},t)$ fields should vanish at any $\mathbf{x}$ as time goes respectively to $T_{\rm min}$ or $T_{\rm max}$ is not satisfied since at these times

$$B_\mu(\mathbf{x},t) \sim a_\mu(r,t) \sim \epsilon^{\mu\nu}\partial_\nu\psi(r,t)/\rho^2(r,t) \ , \tag{3.35}$$

which does not vanish since the ψ-pulses move *undistorted* and do not tend to zero at large early or late times. Here we differ from the claim made in Ref. [10] that the amplitude of ψ-pulses vanishes at early and late times. This claim of Ref. [10] (which mistakenly quotes Ref. [8] for the justification of the claim) contradicts to the arguments stated earlier in the Section as well as to the arguments of Ref. [8] and the Fig. 2.

In order to apply the formalism of Section 2 to the case of classical fields (3.31)-(3.34) in the background, the background (3.31)-(3.34) should be modified at the early past and the far future to switch off the interaction of the gauge fields with the fermions. This will be done now by switching off the gauge invariant degrees of freedom, ψ and $\rho^2 - 1$, of the background field (3.31)-(3.34) at early times, $t:\ T_{\rm min} \le t < t_*$, and late times, $t:\ t_* < t < T_{\rm max}$,

$$\psi \to 0, \quad \rho^2 - 1 \to 0 \quad \forall t:\ \{t < T_{\rm i} < t_*\} \cup \{t > T_{\rm f} > t_*\} \ . \tag{3.36}$$

The fermion number violation which occurs in such modified classical backgrounds is given by eq. (2.40),

$$\langle \hat{N}_{\rm f} - \hat{N}_{\rm i} \rangle = \nu[U_{\rm out}] - \nu[U_{\rm in}] = \nu[U_{\rm f}] = \sum_n degree(\varphi(r_{*n}, t_{*n})) \ , \tag{3.37}$$

and is independent on the way of how the interaction is switched off at early and late times and neither it depends on the times $T_{\rm i}$ and $T_{\rm f}$ as far as their absolute values are much greater than $\max_n |t_{*n}|$.

The procedure described by (3.36) corresponds to the situation of interest where an initial coherent gauge field configuration was produced in the course of quantum collision at some early time, $T_{\rm i}$, and then evolved classically before decaying into quantum radiation at some late time, $T_{\rm f}$. In this work we are interested in the violation of the fermion number which occurred during the classical evolution of the initial coherent state before it decayed. We assume here that there was no fermion number violation before the coherent field was created or after it decayed.

The fermion number in our approach is violated only during the classical evolution of the initial coherent configuration and not at the moment of its creation or decay. Equation (3.37) establishes a selection rule for fermion number violation in the background of a classical solution in the spherical ansatz: *the change of the numbers of fermions is equal to the sum of the degrees of φ in each singular point.* This is an integer by construction while the topological charge Q is not [7-8].

As an example we consider a special case of a Lüscher-Schechter solution with $\tau_0 = 1$ and $\varepsilon = 1$ depicted on Figs. 1 and 2. This solution has a single singular point with the degree of φ being equal to unity. Thus, the violation of the fermion number in the presence of this solution is one, while its topological charge is non-integer [7].

4. Discussion

In this work we considered fermion number violation in the background of a pure $SU(2)$ gauge field in Minkowski space using the method of N. Christ [9] reviewed in Section 2. Then the method was applied for the case of classical solutions in the spherical ansatz in the background. Fermion number violation in such backgrounds was considered also in the past in Refs. [8] and [10]. We will first compare our results to the interpretation of Ref. [8]. Naively applying the anomaly equation (1.2), the net number of fermions produced

was interpreted in Ref. [8] as to be given by a topological charge Q of the solution in the background. Since Q is non-integer in general, the violation of the fermion number being equal to Q was treated in Ref. [8] in a quantum average sense. That is, in every experiment the violation is an integer, but averaging over the experiments one can obtain a non-integer result according to [8]. We no longer believe in this conclusion. In the present approach we switch off the gauge invariant degrees of freedom of the background at early and late times assuming that the classical configuration did not exist forever, but was created at some early time and decayed into quantum radiation at some late time. This allowed us to treat fermions as free in the early past and late future. We believe that the question of the fermion production is not well defined for the classical background which existed forever, since the fermions are never free in this case and the particle interpretation is a conceptual difficulty in this case.

Similarly to our work, the Christ's approach [9] was also used in Ref. [10] to calculate the violation of the fermion number in presence of the classical solutions in the spherical ansatz. We do not agree, however, with the method of Ref. [10] which relied on an incorrect assumption that the ψ field was vanishing at early and late times and, thus, the method of Ref. [10] of handling the Christ's formalism cannot be applied to the case of the classical solutions.

It is also rather instructive to compare our result with the result of Ref. [15] where the violation of the fermion number was studied in the gauge theory in the Higgs phase. It was shown there that the number of fermions produced is equal to the change of the winding number of the Higgs field. In our approach we do not have a fundamental Higgs field, but a Higgs-like field χ, eq. (3.7), appears in the spherical ansatz which is a Higgs field of the (1+1)-dimensional Abelian Higgs model, eq. (3.3). Our selection rule then implies that the change of the numbers of fermions is equal to the change of the winding number of the Higgs-like field χ which looks somewhat parallel to the result of Ref. [15]. Nevertheless, the method of Ref. [15] relies on the existence of a gap in the fermion spectrum and cannot be applied to our theory with massless fermions.

But the method of Section 2 *can* be used for the case of the background being a classical solution of the gauge theory in the Higgs phase. Applying the Christ's approach to this case we would readily reproduce the result of Ref. [15]. This was done in Ref.

[16]. In fact, the case of the Higgs phase is easier than the case of the pure gauge theory since the classical solutions in the Higgs phase dissipate at early and late times because the gauge field becomes massive. In the background of dissipating classical solutions fermions do become free in the early past and the late future and the gauge background does not have to be switched off. In this case the violation of the fermion number is always integer [15-16] and does not have much to do with the topological charge Q which in this case is not even well defined [15]. This means that one should be rather careful with a naive interpretation of the anomaly equation. In the Christ's approach [9] discussed in Section 2 this difficulty is avoided by introducing the "fermion" charges of the radiating gauge fields $q^{\rm out}$ and $q^{\rm in}$ which have nothing to do with the actual number of fermions and have to be subtracted from the right hand side of the anomaly equation, see eq. (2.39).

The necessary condition for fermion number violation in our approach is the vanishing of the ρ^2 field. For the perturbative solutions of Ref. [8] the field ρ^2 never vanishes and the barion number violation is zero. (This point was already addressed in Ref. [10].) Thus, the non-zero fermion number violation cannot be achieved in the framework of perturbation theory even in the model with an unbroken gauge group. The non-perturbative solutions [7] of the Section 3 behave like $1/g$ with the energy, eq. (3.13), $E \sim 1/g^2$. This implies that even in the case of a QCD-like theory the classical field which causes chiral fermion number violation can be constructed only from the number of initial particles of order of $1/\alpha$. This suggests that there is a sphaleron-like configuration which is the top of the barrier separating vacua with different chiral fermion numbers even in QCD. Of course, QCD is a scale-invariant model on the classical level and the top of the barrier depends on an arbitrary scale which is supposingly fixed by quantum effects. In Section 3 we saw that the violation of the fermion number occurs when the classical field passes through the "singular" point where $\rho^2 = 0$. For the specific case of Lüscher-Schechter solutions fermion number is violated by ± 1 when the field q of the associated mechanical problem passes over the top, $U = 1/2$, of the double well potential, $U = \frac{1}{2}q^2(\mathrm{q}+2)^2$. We associate the configuration (3.16) with $q = -1$,

$$\rho^2 = \sin^2 w, \quad \psi = 0, \tag{4.1}$$

with a sphaleron-like configuration in QCD. The configuration (4.1) is, in fact, a classical (time-dependent) solution of de Alfaro, Fubini and Furlan [17]. The interpretation of this

solution as an exploding sphaleron of QCD was recently made in Ref. [18].

If the quantum effects of QCD fix the scale, then some quantum analog of de Alfaro-Fubini-Furlan solution will become a real quantum sphaleron of QCD and for the energy below the quantum sphaleron mass the chirality violation for massless fermions will not occur, while for energies higher than this mass the violation may very well happen.

Acknowledgments

I am grateful to Eddie Farhi and Bob Singleton for infinite discussions about classical solutions. I would like to thank N. Christ, J. Goldstone and N. Manton for discussions on anomalous fermion number violation. I also benefited from T. Gould, S. Hsu and K. Rajagopal and from early discussions with V.A. Rubakov and L. Yaffe on fermion number violation.

REFERENCES

[1] G. 't Hooft, *Phys. Rev.* **D14** (1976) 3432; (E) **D18** (1978) 2199.

[2] S. Adler, *Phys. Rev.* **177** (1969) 2426;
J. Bell and R. Jackiw, *Nuovo Cimento* **51** (1969) 47.

[3] N. Manton, *Phys. Rev.* **D28** (1983) 2019;
F. Klinkhamer and N. Manton, *Phys. Rev.* **D30** (1984) 2212.

[4] A. Belavin, A. Polyakov, A. Schwarz and Yu. Tyupkin, *Phys. Lett.* **B59** (1975) 85.

[5] A. Ringwald, *Nucl. Phys.* **B330** (1990) 1.

[6] M. Mattis, *Phys. Rep.* **214** (1992) 159;
P.G. Tinyakov, *Int. J. Mod. Phys.* **A8** (1993) 1823.

[7] E. Farhi, V.V. Khoze and R. Singleton, *Phys. Rev.* **D 47**, 5551 (1993).

[8] E. Farhi, V.V. Khoze, K. Rajagopal and R. Singleton, *Phys. Rev.* **D 50**, 4162 (1994).

[9] N. Christ, *Phys. Rev.* **D21** (1980) 1591.

[10] T.M. Gould and S.D.H. Hsu, Harvard preprint HUTP-94/A036, hep-ph/9410407.

[11] R. Jackiw and K. Johnson, *Phys. Rev.* **182** (1969) 1459.

[12] E. Witten, *Phys. Rev. Lett.* **38** (1977) 121;
B. Ratra and L.G. Yaffe, *Phys. Lett.* **B205** (1988) 57.

[13] M. Lüscher, *Phys. Lett.* **B70** (1977) 321;
B. Schechter, *Phys. Rev.* **D16** (1977) 3015.

[14] S. Wolfram, *Mathematica, A System for Doing Mathematics by Computer*, 2^{nd} Edition, Addison Wesley 1992.

[15] E. Farhi, J. Goldstone, S. Gutmann, K. Rajagopal and R. Singleton, MIT preprint CTP 2370, hep-ph/9410365.

[16] T.M. Gould and S.D.H. Hsu, Harvard preprint HUTP-94/A039, hep-ph/9410408.

[17] V. de Alfaro, S. Fubini and G. Furlan, *Phys. Lett.* **65B** (1976) 163.

[18] G. Gibbons and A. Steif, Cambridge preprint DAMTP/R94/53.

Figure Captions

Fig. 1 The ρ^2-component of a Lüscher-Schechter solution with $\varepsilon = 1$ and $\tau_0 = 1$. Figure 1a shows the (incoming) r-profiles of ρ^2 for a sequence of negative times: $-10 < t < 0$. Figure 1b shows the (outgoing) r-profiles of ρ^2 for a sequence of positive times: $0 < t < 10$.

Fig. 2 The ψ-component of a Lüscher-Schechter solution with $\varepsilon = 1$ and $\tau_0 = 1$. Figure 2a shows the (incoming) r-profiles of ψ for a sequence of negative times: $-10 < t < 0$. Figure 2b shows the (outgoing) r-profiles of ψ for a sequence of positive times: $0 < t < 10$.

Fig. 3 Contours of integration, $\mathcal{C}_{(r,T_{\rm min})\mapsto(r,t)}$, used in eq. (3.30) to define a continuous gauge transformation are shown for two cases: 1) $r \equiv r_1 < r_*$; and 2)$r \equiv r_2 > r_*$.

Fig. 1a

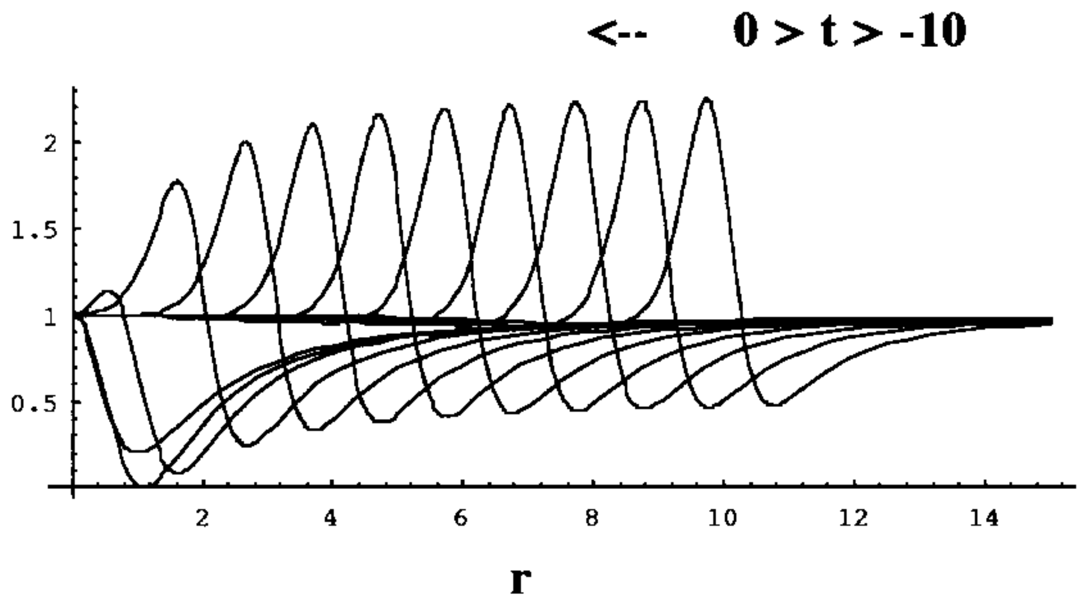


Fig. 1b

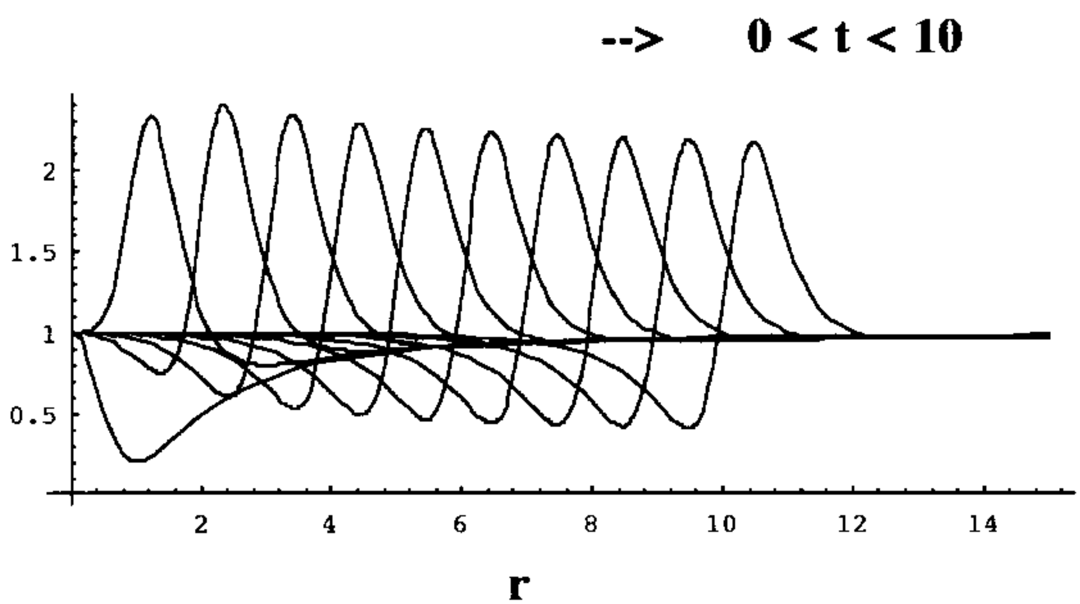
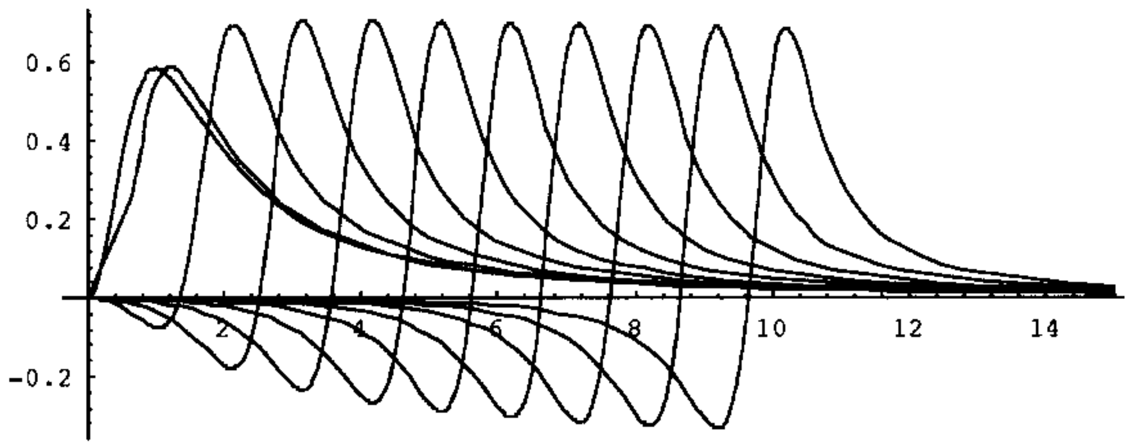


Fig. 2a

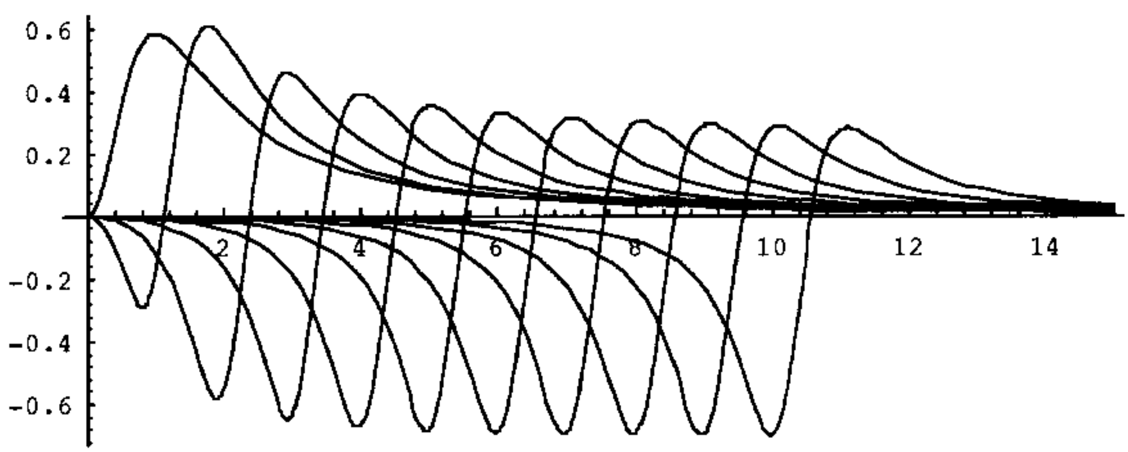
<-- $0 > t > -10$



r

Fig. 2b

--> $0 < t < 10$



r